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Calculus III Project I Report

Introduction

We captains, Clement, Quinn, and Thomson, of the USS Marty McFly, are space pirates who stole the immensely valuable 1.21 GW flux capacitor from the scientists on planet Docbrown. We realized that we were undertaking a dangerous mission, and that we needed to be prepared in the event that we were discovered. Therefore, we chose the ideal escape path from the surface. The path we took is given by:

$$r(\mathrm{t}) = \{\sin(2\pi t), t^3, \cos^2(2\pi t)\} \quad \text{Eq. 1}$$

Realizing that we might be discovered and come under attack, we analyzed our path and the paths of interplanetary missiles to determine whether or not we would be in danger. Our ship's supercomputer, *Mathematica,* determined the important components of our path: velocity, TNB vectors, arc length, curvature, and torsion. Our technology aided us in evading capture and returning home safely with the flux capacitor.

Analysis of Escape Trajectory

To help us escape, our supercomputer analyzed the following:

Escape Flight Path

The equation of our path is given by:

$$r(\mathrm{t}) = \{\sin(2\pi t), t^3, \cos^2(2\pi t)\} \quad \text{Eq. 1}$$

To help us visualize our path, our supercomputer plotted our path from the planet for the first two minutes. Our starting location is indicated by a green dot, and our location at 2 minutes is indicated by a red dot.

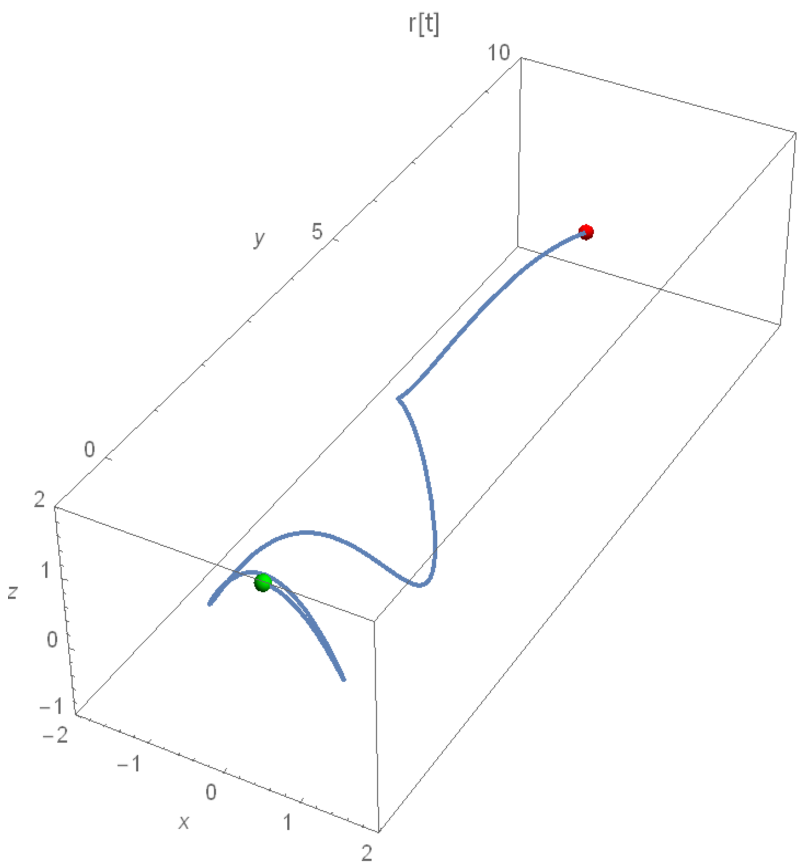

Figure 1: Plot of the path of the USS Marty McFly

Spacecraft Velocity

By taking the derivative of our path with respect to time, we obtained the equation for the velocity of our spacecraft.

$$v(\mathrm{t}) = r'(t) \quad \text{Eq. 2}$$

Our supercomputer created the following plot of our velocity vector for the first two minutes.

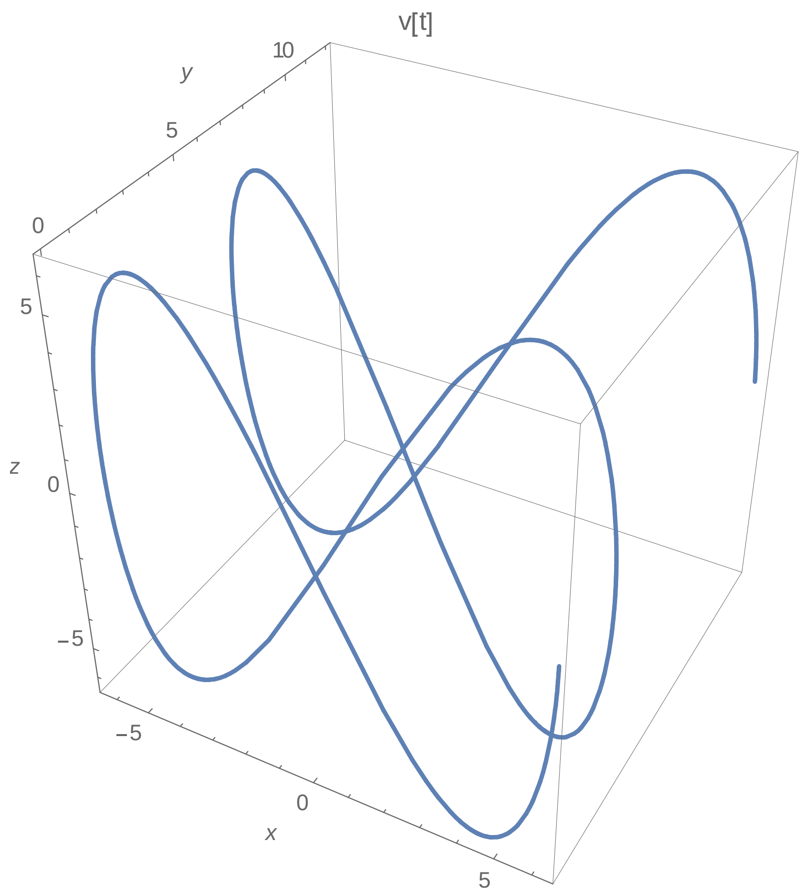

Figure 2: Plot of USS Marty McFly's velocity as a function of time

Tangential vector

Our supercomputer then calculated the unit tangent vector of our craft. This vector shows the direction of our path as a function of time. This was calculated using the equation:

$$T(\mathrm{t}) = \frac{\mathrm{v}(t)}{|v(t)|} \quad \text{Eq. 3}$$

Our supercomputer plotted the unit tangent vector over the first four minutes of our flight, yielding the following graph:

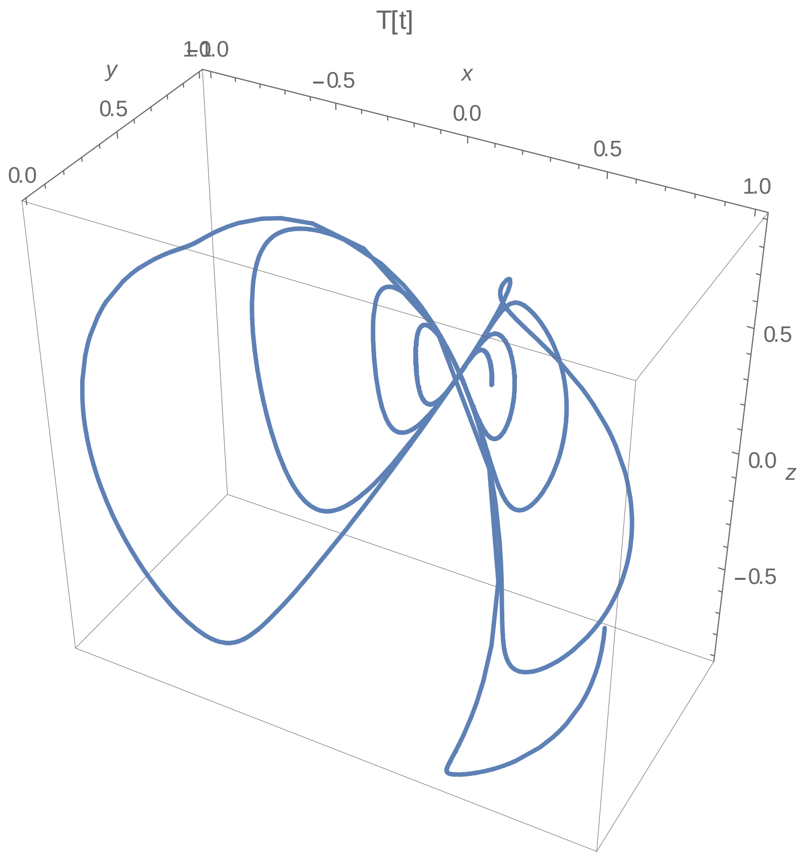

Figure 3: Plot of the USS Marty McFly's unit tangent vector as a function of time

The tangential path exists on a sphere, given by the equation:

$$x^2 + y^2 + z^2 = 1 \quad \text{Eq. 4}$$

The reason for this is that, by definition, any unit vector, including the unit tangent, has a magnitude of 1. The magnitude of the vector is given by:

$$\sqrt{x^2 + y^2 + z^2} = 1 \quad \text{Eq. 5}$$

Squaring both sides yields the equation for a sphere, as shown by Equation 4.

The arc length of the path

Our supercomputer then determined the arc length of the path as a function of time, using the equation:

$$S(t) = \int_0^t |v(t)|dt \quad \text{Eq. 6}$$

Based on this equation, our supercomputer plotted the arc length over the first two minutes of flight:

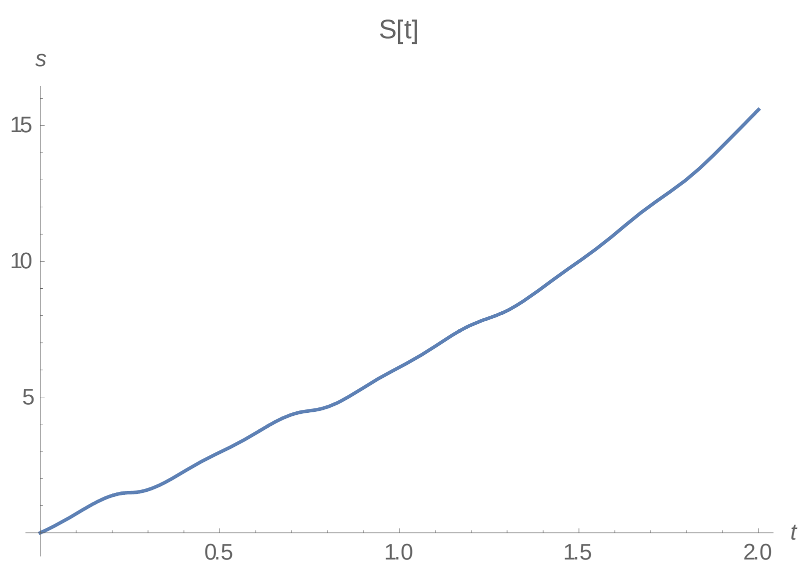

Figure 4: Plot of arc length of the path as a function of time

The total distance that we traveled during the first two minutes of our escape can be calculated using Equation 6:

$$S(2) = 15.5842 \; distance \; units$$

The normal vector

The supercomputer then determined the normal vector of our path as a function of time.

This vector shows us the direction that our ship's path curved at any point.

This vector is given by:

$$n(\mathrm{t}) = \frac{T'(t)}{|\mathrm{T(t)}|} \quad \text{Eq. 7}$$

Our supercomputer then plotted the unit normal vector over the first two minutes of our flight, yielding the following graph:

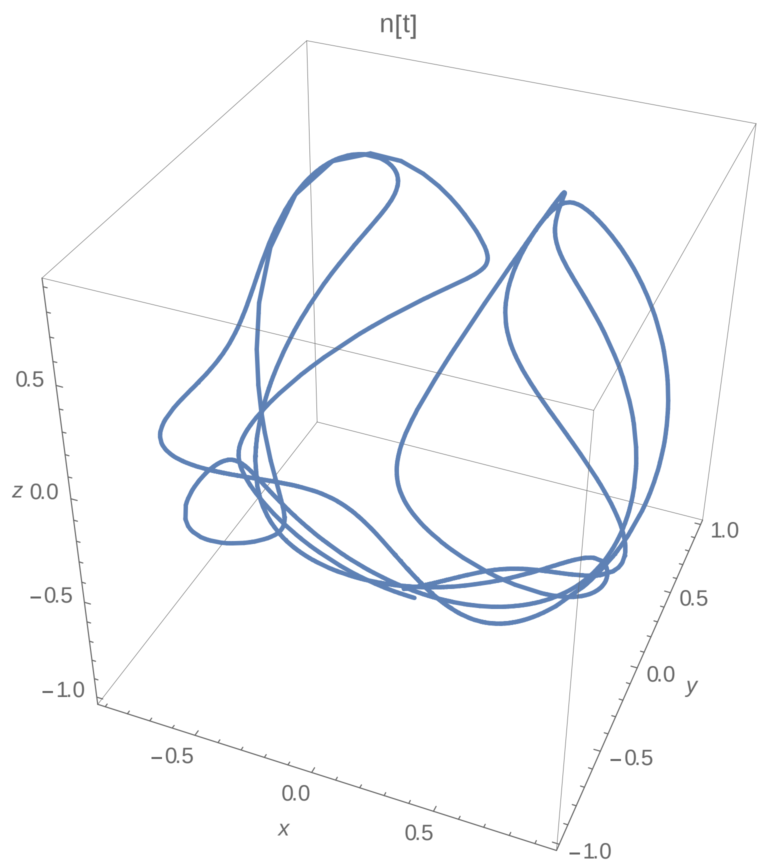

Figure 5: Plot of the USS Marty McFly's normal vector as a function of time

The normal vector to our spacecraft's path is also a unit vector bounded on the sphere given by equation 4. The proof for this is identical to that of the tangent vector.

Binormal Vector

Our supercomputer then calculated the unit binormal vector of our craft. This vector is orthogonal to both the tangential and normal vectors for all times. The binormal vector therefore functions as the normal vector to the plane spanned by the tangential and normal vectors. The binormal vector was calculated using the equation:

$$B(\mathrm{t}) = T(t) \times n(t) \quad \text{Eq. 8}$$

Our supercomputer then plotted the unit binormal vector over the first two minutes of our flight, yielding the following graph:

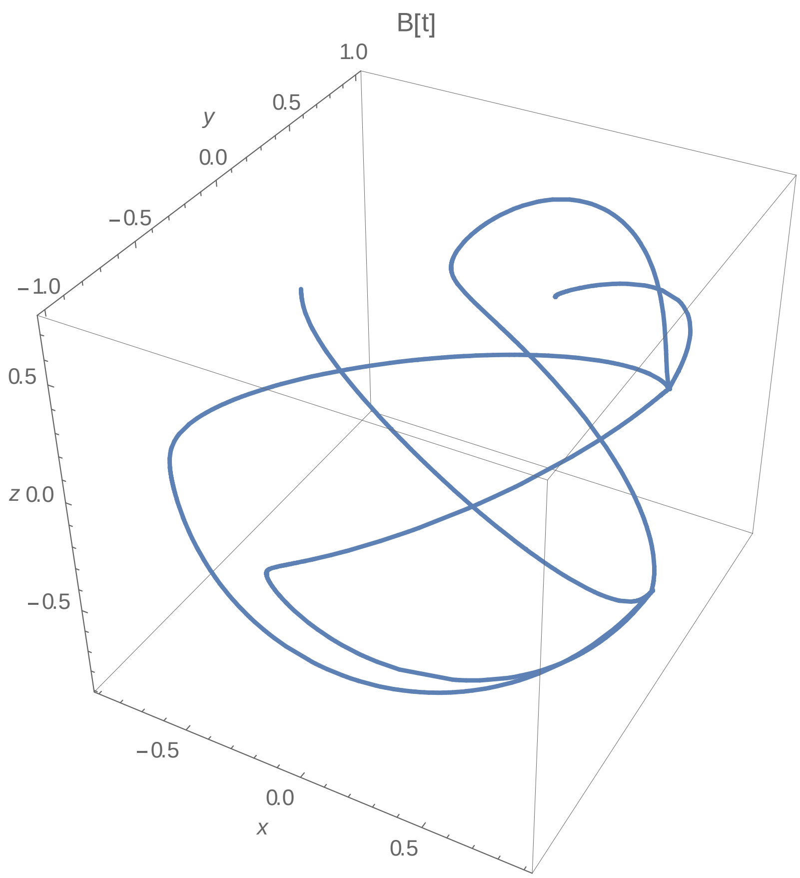

Figure 6: Plot of the USS Marty McFly's binormal vector as a function of time

Like the tangential and normal vectors, the binormal vector is a unit vector that lies on the sphere given by Equation 5.

Curvature of the path

As time gets large, the curvature must approach zero due to the path taken seen in Equation 1:

$$r(\mathrm{t}) = \{\sin(2\pi t), t^3, \cos^2(2\pi t)\} \quad \text{Eq. 1}$$

The x and z components range from [-1,1] and [0,1] respectively. In contrast, as time approaches infinity, the rate at which the y-component grows will also go to infinity. Therefore, as time approaches infinity, the path will become a straight line. A straight line has zero curvature, and thus, the curvature of the ship's path will also become zero as time becomes very large.

Our supercomputer then calculated the curvature of our craft's path. The curvature was calculated using the equation:

$$\kappa(\mathrm{t}) = \frac{|T'(t)|}{|v(t)|} \quad \text{Eq. 9}$$

Our supercomputer then plotted the curvature of the path over the first ten minutes of flight:

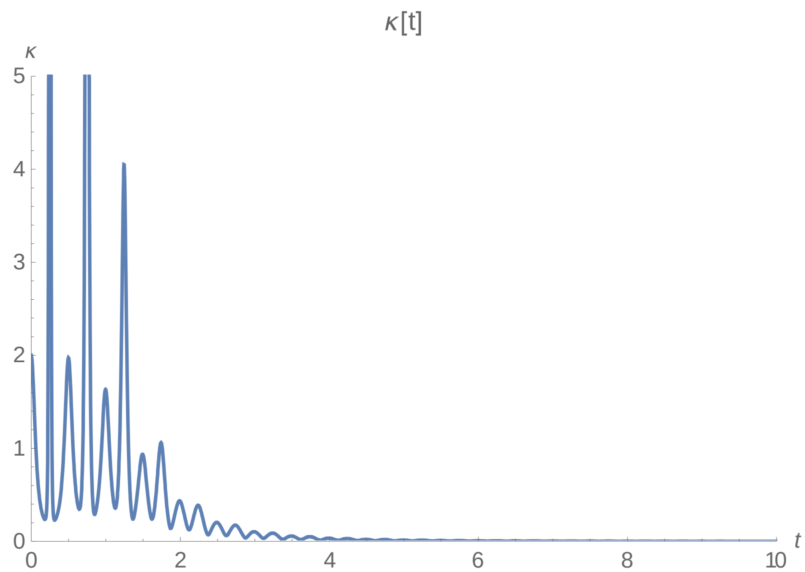

Figure 7: Curvature of the USS Marty McFly's path as a function of time

The torsion of the path

As time approaches infinity, over the time it takes for the binormal vector to change, the arc length has increased by a near-infinite amount; therefore, the change in the binormal with respect to arc length approaches 0. The torsion must also approach 0 as time becomes infinitely large.

Our supercomputer then calculated the torsion of the path. The torsion was calculated using the equation:

$$\tau(\mathrm{t}) = -\frac{B'(t)}{|v(t)|} . n(t) \quad \text{Eq. 10}$$

Our supercomputer then plotted the torsion of the path over the first twenty minutes of flight:

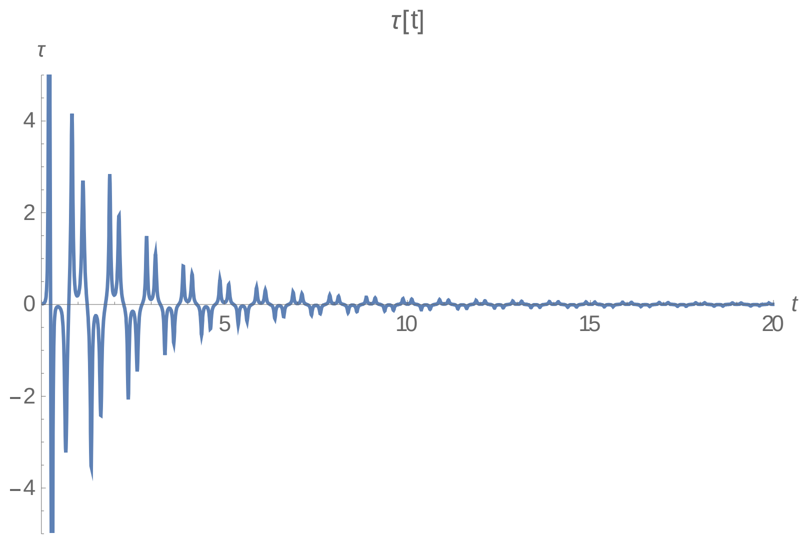

Figure 8: The torsion of the USS Marty McFly's path as a function of time

An Explosive Development

The missile's planned path

Unfortunately, one minute after we stole the flux capacitor, the citizens of Docbrown launched a missile that follows the following path

$$r_{m1}(\mathrm{t}) = \{1.2\sin(2\pi t) + 0.3, t^4, 1.1\cos^2(2\pi(t + 0.2))\} \quad \text{Eq. 11}$$

The missile would explode if it got within 0.5 distance units of the spacecraft. Our supercomputer calculated the minimum distance between the missile and our spacecraft after the missile was fired from the surface of Docbrown. The supercomputer used the following equation to find the distance between the missile and the spacecraft:

$$\text{Distance}_1(\mathrm{t}) = |r(t) - r_{m1}(\mathrm{t})| \quad \text{Eq. 12}$$

In order to find where distance was at a minimum, the supercomputer set the derivative of the distance equation equal to 0 to find the critical points. The supercomputer then evaluated each of the critical using equation 12 to find the minimum distance. By comparing the distance at each critical point and the distance when the missile was launched (at t=1 minute) and when time approaches infinity, the supercomputer was able to determine the absolute minimum distance. The minimum distance between the spacecraft and the missile was 0.506 distance units at time t= 1.142 seconds. Since 0.507 distance units> 0.5 distance units, the supercomputer concluded that the missile would not explode.

The missile's actual path

Unfortunately, the actual path of the missile was different than the planned path of the missile. The supercomputer calculated the actual path as the following equation:

$$\mathrm{r}_{m2}(\mathrm{t}) = \{\sin(2\pi t) + 0.4, t^4, \cos^2(2\pi(t + 0.2)) \quad \text{Eq. 13}$$

The supercomputer used the following equation to find the distance between the missile and the spacecraft:

$$\mathrm{Distance}_2(\mathrm{t}) = |r(t) - r_{m2}(\mathrm{t})| \quad \text{Eq. 14}$$

To find where the distance was at a minimum the supercomputer once again set the derivative of the distance equation equal to 0 to find the critical points. The supercomputer then evaluated each of the critical using equation 14 to find the minimum distance. By comparing the distance at each critical point and the distance when the missile was launched (at t=1 minute) and when time approaches infinity, the supercomputer was able to determine the absolute minimum distance. The minimum distance between the spacecraft and the missile was 0.459 distance units at time t= 1.147 seconds. Since 0.459 distance units<0.5 distance units, the missile exploded.

Conclusion

Our supercomputer's analysis of our path determined that, initially, our path is erratic with high curvature and torsion, making us very hard to follow. In the long run, our path would have nearly become a straight line, with curvature and torsion approaching zero. Unfortunately, our short term evasive maneuvers were not enough. Although the predicted path of the missile would have barely missed us, its actual path came right for our ship. Luckily, we were able to activate the flux capacitor in time, transporting us and our ship back to 1955. Yar!