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Calculus III Project III Report

Introduction:

We are Cram, Nayr, and Rehpotsirhc, of the Yacht-consulting firm Snoitairav Suluclac. We hear that you, Captain Sirron Mada, are competing in the annual elliptical race around Suluclac, and would like our help. We have taken a look at your old square-rigged vessel, and have analyzed its potential for this race. We have also run analyses on the now standard, and far superior albeit expensive type of vessel, the racing yacht. We have also calculated the optimum route around the island, Suluclac, for this vessel based on the historic wind patterns of the region. Below are our results.

The Square-Rigged Vessel:

The square-rigged vessel is designed to go in the same direction as the wind. We can define a function that relates the velocity of the vessel at any point to the wind speed and the angle between the vessel and the wind.

$$\sigma_S = |\boldsymbol{w}|\max\{10(\cos(\theta) - 0.9), 0\} \text{ Eq. 1}$$

We can see that this function can be written as

$$\sigma_S = |\boldsymbol{w}|\mathrm{g}(\theta), \text{where } \mathrm{g}(\theta) = \max\{10(\cos(\theta) - 0.9), 0\} \text{ Eq. 2}$$

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Below we have graphed the function $g(\theta)$ for angles between -π and π

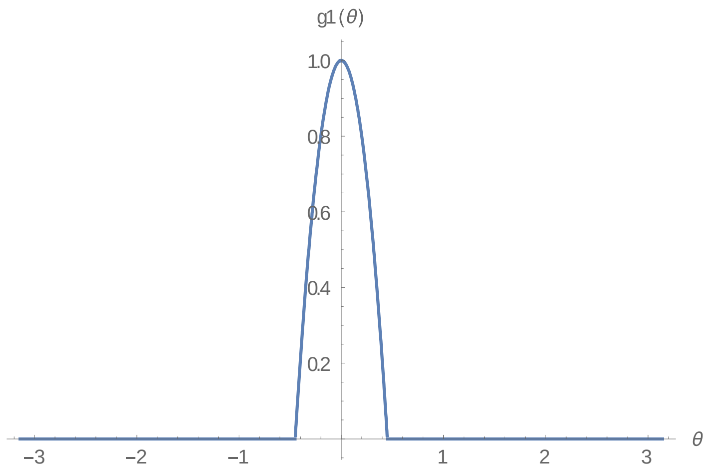

Figure 1: $g(\theta)$ for the square-rigged yacht

Although this function is defined for all angles between -π and π, it is only nonzero for a narrow range about the origin. This means that the ship must be going nearly parallel to the wind in order to move. The angle from the direction of the wind must range between -0.451 and 0.451 radians in order to have a nonzero velocity. The function is symmetric about the origin, indicating that the function is even. In the real world, this means that it doesn't matter whether the ship is turned toward starboard or port. As long as the absolute value of the angle is the same, the resulting velocity will be the same.

Now that we have the information about the square-rigged ship's velocity, we can determine the optimal path around the island. The path must be elliptic and begin 10 miles east of the center of the island, loop counterclockwise, pass through the point 10 miles west of the island, and continue counterclockwise back to the start. Below is an equation for this path, with distance measured in 10s of miles, and u being an arbitrary parameter:

$$\boldsymbol{r}\big(x(u,b),y(u,b)\big) \text{ where } x(u,b) = \cos(u)\,, y(u,b) = bsin(u) \text{ Eq.3}$$

Our aim is to find an expression for σ_S along the path. The definition of the dot product of w and T, where t is a unit vector parallel to the path is:

$$\boldsymbol{w} \cdot \widehat{\boldsymbol{T}} = |\boldsymbol{w}|\cos(\theta) \text{ Eq.4}$$

We can then rewrite σ_S as a function of u and b. We will need w and T in order to evaluate this.

$$\sigma_S(u,b) = \max\{10(\boldsymbol{w} \cdot \widehat{\boldsymbol{T}}\text{-}0.9|\boldsymbol{w}|),0\} \text{ Eq.5}$$

We can derive $\widehat{\boldsymbol{T}}$ by differentiating the path vector **r**(u,b) with respect to u to get **v**(u,b) and dividing by its magnitude. This vector is omitted for simplicity's sake.

Based on our in-depth research, we have determined the wind patterns around the island. Below is our model of the wind at any point:

$$\boldsymbol{w} = -\frac{x^2+y}{1+x^2+y^2}\hat{\boldsymbol{\imath}} + \frac{x-y^2}{1+x^2+y^2}\hat{\boldsymbol{\jmath}} \quad \text{Eq. 6}$$

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By evaluating w along the path and plugging w and T into the equation, we can derive an expression for $\sigma_s(u,b)$. Below is a plot of σ_b on the first and second legs of the race, respectively.

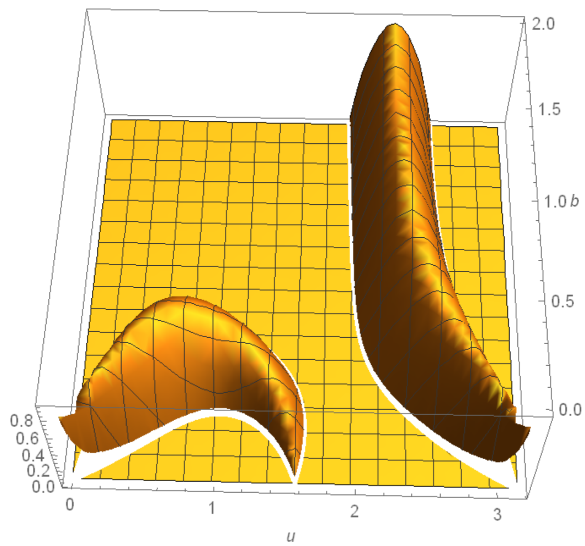

Figure 2: $\sigma_s(u,b)$. for the first leg of the race

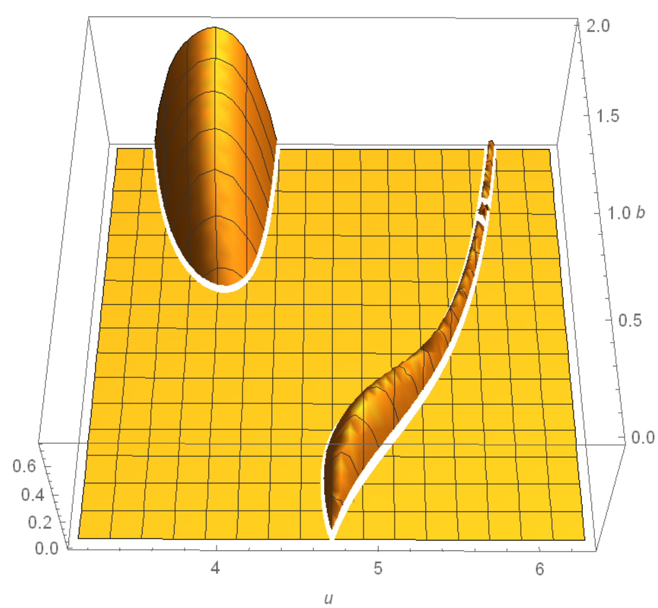

Figure 3: $\sigma_s(u,b)$. for the second leg of the race

As you can see, for every possible value of b between 0.01 and 2, the velocity will be zero for much of the path. We regret to inform you that you will not be able to use this vessel to complete the race. Even if the path is not elliptical (and we would remind you that this is strictly against the rules), we cannot find a path for you to complete this race. You will either crash into the island or be stranded at sea. You would be able to complete the first leg of the race following a non-elliptical path; however, no path will allow you to finish the second half of the race, as your vessel will be swept away from the island. Below is a graph of wind at any point around the island as well as a possible path for your boat so as to finish the first leg.

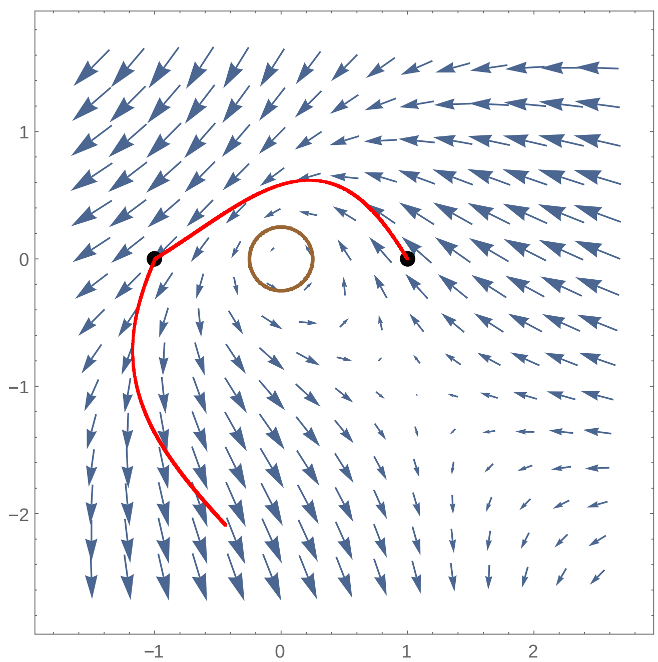

Figure 4: wind velocities near the island. The black dots show the start and midpoint of the race. The red line indicates a potential path for the boat.

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Exploring Another Option:

Fortunately for you, we partner with a firm that sells yachts, Ekatusay Yachts. They have a reasonably priced modern racing yacht that could be perfect for this race. Her name is *Noitalucirc,* and she's just what you need to succeed. She will be able to operate at a much wind range of angles from the wind than your other vessel. We can define a similar function, σ_y that relates wind speed at any point to angle from the wind:

$$\sigma_y = |\boldsymbol{w}| \frac{1+\cos(\theta)}{2} \quad \text{Eq. 7}$$

Again, this function can be written as

$$\sigma_y = |\boldsymbol{w}| g(\theta), where\ g(\theta) = \frac{1+\cos(\theta)}{2} \quad \text{Eq. 8}$$

Below is a graph of $g(\theta)$ for values of θ between −π and π

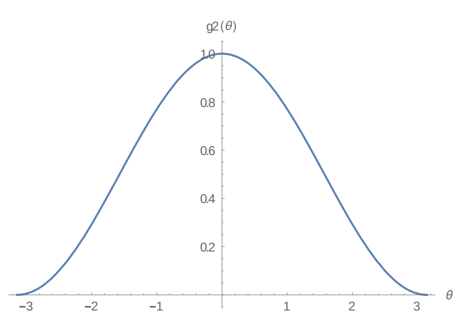

Figure 5: $g(\theta)$ for the modern-racing yacht

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This function is nonzero for all values of θ between -π and π. Therefore, there is a wider range of angles where the velocity is nonzero. The function is symmetric about the origin, indicating that the function is even. In the real world, this means that it doesn't matter whether the ship is turned toward starboard or port. As long as the absolute value of the angle is the same, the resulting velocity will be the same. This model is a bit imprecise because the yacht performs best at an angle from the wind, while this model shows it working best parallel to the wind. Still, this model is adequate to show this vessel's superiority to the square-rigged vessel.

The path and the wind speed are defined exactly the same as with the square-rigged vessel. Below is σ_y with $\boldsymbol{w}$ and $\hat{T}$ plugged in:

Below is a plot of σ_y on the first and second legs of the race, respectively.

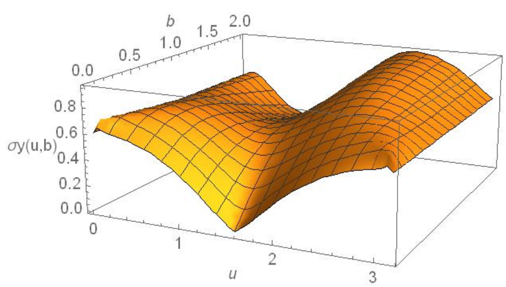

Figure 6: $\sigma_y(u, b)$ for the first leg of the race

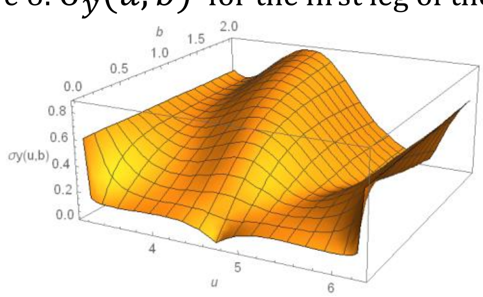

Figure 7: $\sigma_y(u, b)$ for the second leg of the race

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Since this vessel has a large range in which it can travel without its velocity being nonzero, there will be an elliptical path that it can take around the island to complete the race. The only path that would cause the velocity to be zero would be an ellipse with a b value, the stretch of the elliptical path in the north-south direction, of 0. This would make the path a straight line, and would lead directly through the island. The b value is also restricted by the size of the island. The b value must be larger than 0.25 in order for the path of the vessel to go around the island and not through it. Our goal is to finish the race in the smallest amount of time, meaning we need to find a value b that is nonzero and larger than 0.25, which will minimize the time to complete the path. The elliptical path can be different for each leg of the race, meaning that the b value will be different for each leg of the race. In order to calculate time, we must integrate the inverse of velocity, given by σ_y, over the arc length.

$$\int_C \frac{1}{\sigma(u)} ds \quad Eq. 9$$

This integral can also be rewritten as:

$$\int_C \frac{|v(u)|}{\sigma(u)} du\,, because\ ds = |v(u)|du \quad Eq. 10$$

We then plot how the time to complete the first leg of the race is affected by the b value of the path:

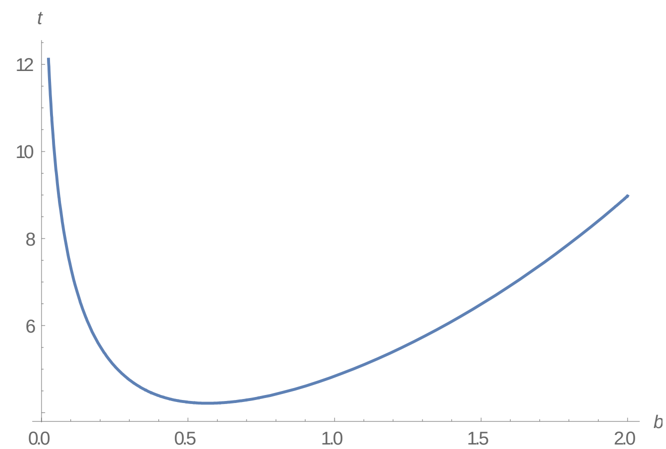

Figure 8: t(b) for the first leg

If conditions remain consistent with our predictions on race day, and we believe that they will, the path we have provided to you, with a b value of 5.67147 miles for the first leg will minimize the time to complete the first leg. We then plot how the time to complete the second leg of the race is affected by the b value of the path:

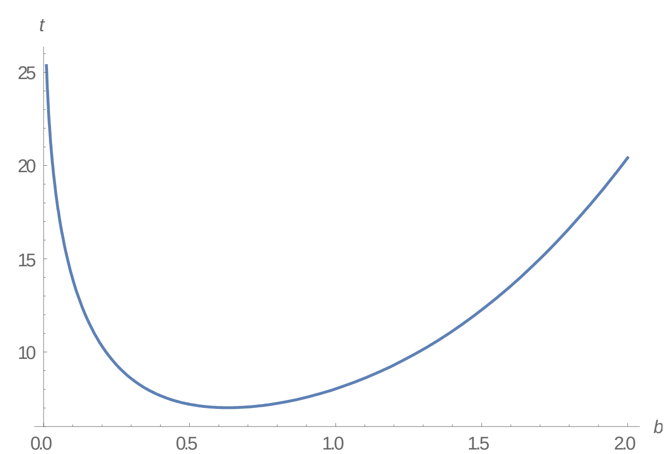

Figure 9: t(b) for the second leg

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The path we have provided to you, with a b value 6.31752 miles for the second leg will minimize the time to complete the second leg. Taking this path, your new yacht will travel 25.0885 miles in leg 1, 25.9589 miles in leg 2, for a total distance of 51.0474 miles.

Below is a graph showing the island, the wind field, and our recommended paths for both the first leg as well as the second:

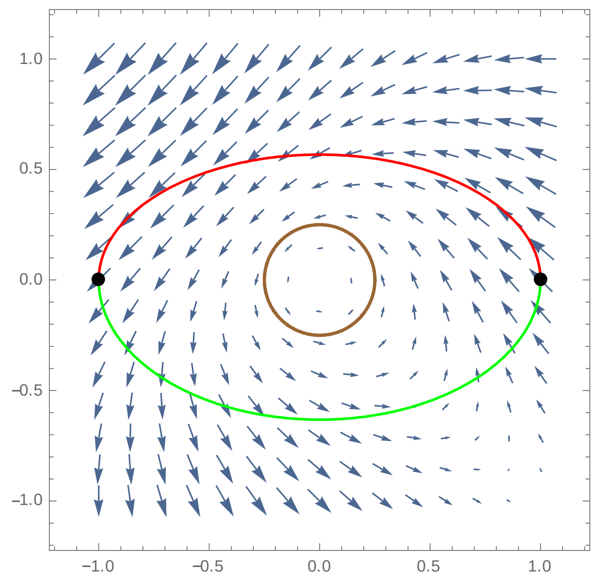

Figure 10: Map of modern-racing yacht path. The red line represents the first leg; the green line represents the second leg. The brown circle is the island.

Following these two paths, the travel time and total distance should be as follows:

	First Leg	Second Leg	Total
Travel Time (hours)	4.21726	7.01502	11.23228
Distance Travelled (miles)	25.0885	25.9589	51.0474

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Conclusion

Unfortunately, your square-rigged vessel will be unable to complete the Elliptical Race around the Pacific island Suluclac. As it requires to be nearly parallel to the wind in order to move, your current boat will either be driven aground or sent far into the ocean.

Investing in the *Noitalucirc*, however, will provide you with the ship needed to not only complete the race, but do so in minimum time. With its ability to catch wind at a wide range of angles, it will be able to steer clear of the island and execute the turns that the square-rigged vessel simply could not. Following the path that we have predicted, you will spend a total of 11.232 hours aboard the *Noitalucirc* for a total distance of 51.047 miles. Thank you for choosing your local yacht consulting firm, Snoitairav Suluclac.